MATH 105A and 110A Review: The determinant and invertibility of the Jacobian

Facts to Know:

The determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

The determinant of a 3×3 matrix is

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Any $n \times n$ matrix A is said to be **invertible** if there exists a matrix B such that

$$AB = I = BA \qquad A^{-1} = B$$

Any $n \times n$ matrix A is invertible if and only if $\det A \neq 0$

The inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

To find the inverse of A, consider the augmented matrix $[A|I]$ and row reduce it to reduced echelon form:

$$[A \mid I] \rightarrow \text{row reduce} \cdots \rightarrow [I \mid A^{-1}]$$

Examples:

$F: \mathbb{R}^2 \to \mathbb{R}^2$

1. Find the inverse of the Jacobian matrix of $F(x, y) = (\underset{f_1}{xy^2}, \underset{f_2}{2xy})$ when possible.

$$\nabla F = \begin{bmatrix} y^2 & 2xy \\ 2y & 2x \end{bmatrix}$$

∇F is inv. iff $0 \neq \det \nabla F = 2xy^2 - 4xy^2$
$= -2xy^2$

iff
x and y are both not zero.

The inverse is:

$$(\nabla F)^{-1} = \frac{1}{-2xy^2}\begin{bmatrix} 2x & -2xy \\ -2y & y^2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{y^2} & \frac{1}{y} \\ \frac{1}{xy} & \frac{-1}{2x} \end{bmatrix}$$

2. Find the inverse of the Jacobian matrix of $F(x, y, z) = (xyz, 2yz, 3z)$ at the point $(1, 1, 1)$ if possible.

$F: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad f_1 \; f_2 \; f_3$

$$\nabla F = \begin{bmatrix} yz & xz & xy \\ 0 & 2z & 2y \\ 0 & 0 & 3 \end{bmatrix}$$

$$\nabla F(1,1,1) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \frac{1}{2}R_2 \to R_2 \\ \frac{1}{3}R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right] R_1 - R_2 \to R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right] R_2 - R_3 \to R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} \end{array}\right]$$

∇F is inv. at $(1,1,1)$ and

$$(\nabla F)^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{pmatrix}.$$